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Parametric optimum analysis of an irreversible Ericsson cryogenic refrigeration cycle working with an ideal Fermi gas

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Abstract. An irreversible model of an Ericsson cryogenic refrigeration cycle working with an ideal Fermi gas is established, which is composed of two isothermal and two isobaric processes. The influence of both the quantum degeneracy and the finite-rate heat transfer between the working fluid and the heat reservoirs on the performance of the cycle is investigated, based on the theory of statistical mechanics and thermodynamic properties of an ideal Fermi gas. The inherent regeneration losses of the cycle are analyzed. Expressions for several important performance parameters such as the coefficient of performance, cooling rate and power input are derived. By using numerical solutions, the cooling rate of the cycle is optimized for a given power input. The maximum cooling rate and the corresponding parameters are calculated numerically. The optimal regions of the coefficient of performance and power input are determined. Especially, the optimal performance of the cycle in the strong and weak gas degeneracy cases and the high temperature limit is discussed in detail. The analytic expressions of some optimized parameters are derived. Some optimum criteria are given. The distinctions and connections between the Ericsson refrigeration cycles working with the Fermi and classical gases are revealed.

Keywords. Cryogenic refrigeration cycle; irreversibility; quantum degeneracy; performance characteristics; optimal analysis.

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1. Introduction

The optimal performance characteristics of thermodynamic cycles have been intensively analyzed in recent years, covering a wide range from classical to quantum thermodynamic cycles [1–19]. Quantum thermodynamic cycles working with the ideal quantum gases [1,5,15,18,19], spin systems [2,3,6,13], harmonic oscillator systems [8,10,12] and multilevel quantum systems [14,16,17] have become interesting research subjects. These investigations facilitate the understanding of the performance of the cryogenic refrigeration cycles.

According to the theory of classical thermodynamics, the Ericsson or Stirling refrigeration cycle working with an ideal gas may possess the condition of perfect regeneration through the use of a reversible regenerator and has the same performance characteristics as those of a Carnot refrigeration cycle. However, when the temperature of the gas is low enough or its density is high enough, the gas deviates from its classical behavior irrespective of whether the properties of the gas obey Bose–Einstein statistics or Fermi–Dirac statistics [20–22]. Under these circumstances, the quantum degeneracy of the gas will become important, so that the Ericsson cryogenic refrigeration cycle using a quantum gas as the working fluid cannot process the condition of perfect regeneration [18,23,24]. Therefore, it is one of the important tasks in the optimal performance analysis of the Ericsson cryogenic refrigeration cycle to consider the influence of the quantum degeneracy of the working fluid and the irreversibility of heat transfer simultaneously.

In the present paper, we will investigate the optimal performance of the irreversible Ericsson cryogenic refrigeration cycle working with an ideal Fermi gas and consisting of two isothermal and two isobaric processes. The paper is organized as follows. In §2, the irreversible model of the Fermi–Ericsson cryogenic refrigeration cycle operating between two heat reservoirs at constant temperatures T_{H} and T_{L} is established and the expressions of the amounts of heat exchange in the various processes of the cycle are given. In §3, the influence of the inherent regenerative losses on the performance of the cycle is analyzed. The time evolutions of the working fluid in the various processes are calculated. In §4, the general expressions of several important parameters of the cycle such as the coefficient of performance, cooling rate and power input are given. The general performance characteristics of the cycle are revealed. The curve of the optimal relation between the cooling rate and the coefficient of performance is obtained. The optimally operating regions of the cycle are determined. In §5, the optimal performance of the cycle is discussed in detail for several interesting cases. The optimum criteria of some important parameters are obtained. Finally, some conclusions are summed up in §6.

2. An irreversible Fermi–Ericsson refrigeration cycle

An Ericsson cryogenic refrigeration cycle using an ideal Fermi gas as the working fluid is composed of two isothermal and two isobaric processes and may be simply called the Fermi–Ericsson refrigeration cycle. It is operated between the heat sink at temperature T_{H} and the cooled space at temperature T_{L}. In order to improve the performance of the cycle, a regenerator is used in the isobaric processes. The entropy–temperature diagram of the cycle is shown in figure 1, where Q_1 and Q_2 are the amounts of heat exchange between the working fluid and the two heat reservoirs during the two isothermal processes at temperatures T_1 and T_2, and Q_{bc} and Q_{da} are the amounts of heat exchange between the working fluid and the regenerator during two isobaric processes at pressures P_{H} and P_{L}. All heats Q_1, Q_2, Q_{bc} and Q_{da} are positive. Because such a cycle is irreversible, its performance is dependent on the heat-transfer law and the heat conductances. It is often assumed that heat

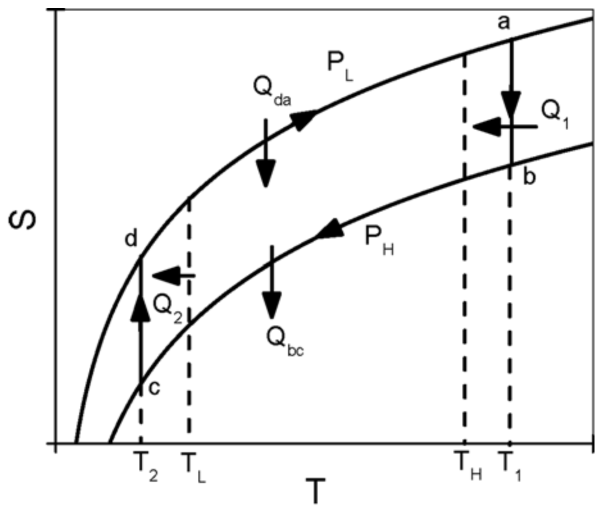

Figure 1. The entropy-temperature diagram of an irreversible Fermi–Ericsson refrigeration cycle.

transfer obeys Newton's law [25], i.e.,

$$Q_1 = \int_0^\tau K_1(t)(T_1 - T_\mathrm{H})\mathrm{d}t, \tag{1}$$

$$Q_2 = \int_0^\tau K_2(t)(T_\mathrm{L} - T_2)\mathrm{d}t, \tag{2}$$

$$Q_{bc} = \int_0^\tau K_3(t)[T(t) - T_\mathrm{R}(t)]\mathrm{d}t \tag{3}$$

and

$$Q_{da} = \int_0^\tau K_4(t)[T_\mathrm{R}(t) - T(t)]\mathrm{d}t, \tag{4}$$

where τ is the cyclic period, $K_1(t)$ and $K_2(t)$ are, respectively, the heat conductances between the working fluid and the two heat reservoirs at temperatures T_H and T_L, and $K_3(t)$ and $K_4(t)$ are the heat conductances between the working fluid and the regenerator, respectively. The relations between the cyclic period τ and these quantities are as follows:

$$K_1(t) = \begin{cases} K_1, & 0 \le t < t_1 \\ 0, & t_1 \le t < \tau \end{cases} \tag{5}$$

$$K_3(t) = \begin{cases} 0, & 0 \le t < t_1 \\ K_3, & t_1 \le t < t_1 + t_3 \\ 0, & t_1 + t_3 \le t < \tau \end{cases} \tag{6}$$

$$K_2(t) = \begin{cases} 0, & 0 \le t < t_1 + t_3 \\ K_2, & t_1 + t_3 \le t < t_1 + t_2 + t_3 \\ 0, & t_1 + t_2 + t_3 \le t < \tau \end{cases} \tag{7}$$

and

$$K_4(t) = \begin{cases} 0, & 0 \le t < \tau - t_4 \\ K_4, & \tau - t_4 \le t < \tau \end{cases} \tag{8}$$

where K_1, K_2, K_3 and K_4 are four constants, and t_1, t_2, t_3 and t_4 are the times spent on processes ab, bc, cd and da, respectively.

According to quantum statistical mechanics, the expressions of the pressure, number density, entropy and heat capacity at constant pressure of an ideal Fermi gas are, respectively, given by

$$P = gkT\lambda^{-3} f_{5/2}(z), \tag{9}$$

$$n = N/V = g\lambda^{-3} f_{3/2}(z), \tag{10}$$

$$S = Nk\left[\frac{5}{2}\frac{f_{5/2}(z)}{f_{3/2}(z)} - \ln z\right] = Nk\left[\frac{5}{2}F(z) - \ln z\right] \tag{11}$$

and

$$C_{\mathrm{P}} = Nk\left[\frac{25}{4}\frac{f_{5/2}^2(z)f_{1/2}(z)}{f_{3/2}^3(z)} - \frac{15}{4}\frac{f_{5/2}(z)}{f_{3/2}(z)}\right] = \frac{5}{2}Nk\frac{\partial}{\partial T}\left[TF(z)\right]_{\mathrm{P}}, \tag{12}$$

where g, k, T, N and V are, respectively, a weight factor that arises from the 'internal structure' of the particle (such as spin), the Boltzmann constant, gas temperature, total number of particles and volume, $z = \exp(\mu/kT)$ and $\lambda = h/(2\pi mkT)^{1/2}$ are, respectively, the fugacity of the gas and the mean thermal wavelength of the particles, μ is the chemical potential of the gas, h is the Planck constant, m is the rest mass of a particle, $F(z) = f_{5/2}(z)/f_{3/2}(z)$ is the specific value of two Fermi–Dirac integrals which is called as the correction function [18,19] and $f_l(z)$ is called the Fermi–Dirac function. It may be expressed as

$$f_l(z) = \frac{1}{\Gamma(l)}\int_0^\infty \frac{x^{l-1}\mathrm{d}x}{z^{-1}\mathrm{e}^x + 1}, \tag{13}$$

where $\Gamma(l)$ is the gamma function [20].

When an ideal Fermi gas is used as the working fluid of the cycle, the amount of heat exchange of the working fluid in various processes of an irreversible Fermi–Ericsson refrigeration cycle may be calculated from eqs (11) and (12) and given by

$$Q_1 = \int_{S_b}^{S_a} T_1 \mathrm{d}S = \frac{5}{2}NkT_1\left[F(z_a) - F(z_b)\right] - NkT_1 \ln(z_a/z_b), \tag{14}$$

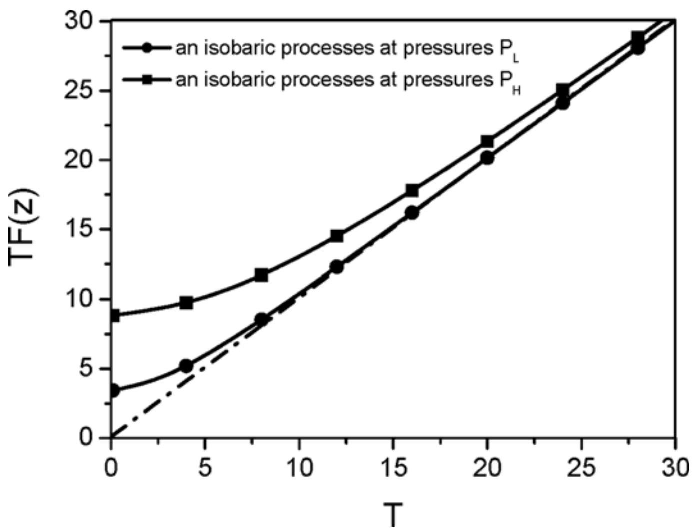

Figure 2. The function $TF(T,P)$ vs. temperature curves for two different pressures. The square and round curves are presented for a constant high pressure ($P_{\mathrm{H}} = 10^7$ Pa) and low pressure ($P_{\mathrm{L}} = 10^6$ Pa), respectively.

$$Q_2 = \int_{S_c}^{S_d} T_2 \mathrm{d}S = \frac{5}{2} NkT_2 \left[F(z_d) - F(z_c)\right] - NkT_2 \ln(z_d/z_c), \tag{15}$$

$$Q_{bc} = \int_{T_2}^{T_1} C_{\mathrm{P}} \mathrm{d}T = \frac{5}{2} Nk[T_1 F(z_b) - T_2 F(z_c)] \tag{16}$$

and

$$Q_{da} = \int_{T_2}^{T_1} C_{\mathrm{P}} \mathrm{d}T = \frac{5}{2} Nk[T_1 F(z_a) - T_2 F(z_d)], \tag{17}$$

respectively, where S_i, z_i and $F(z_i)$ are, respectively, the entropy, fugacity and value of function $F(z)$ of the Fermi gas in state points i ($i = a, b, c, d$). Using eqs (14)–(17), one can calculate the work input per cycle as

$$W = Q_1 - Q_2 + Q_{bc} - Q_{da} = NkT_1 \ln(z_b/z_a) + NkT_2 \ln(z_d/z_c). \tag{18}$$

Using the above equations, we can discuss the optimal performance of a quantum Ericsson refrigeration cycle working with an ideal Fermi gas.

3. Regenerative characteristics and cycle time

From eqs (16) and (17), one can calculate the net amounts of heat transfer between the working fluid and the regenerator during the two isobaric regenerative processes as

$$\Delta Q = Q_{bc} - Q_{da} = \frac{5}{2} Nk\{T_1[F(z_b) - F(z_a)] - T_2[F(z_c) - F(z_d)]\}. \tag{19}$$

In order to expound the characteristics of two regenerative processes in the cycle, we can generate the function $TF(z) = TF(P,T)$ versus temperature T curves for two different pressures by using eqs (9) and (13) and choosing ^{3}He as the working fluid, as shown in figure 2. According to the entropy–temperature diagram of the cycle and the curves in figure 2, one can find the following relation:

$$T_1 F(z_b) - T_1 F(z_a) < T_2 F(z_c) - T_2 F(z_d). \tag{20}$$

Thus, it is seen from eqs (19) and (20) that $\Delta Q = Q_{bc} - Q_{da} < 0$. This implies that the amount of heat exchange Q_{bc} transferred into the regenerator in one regenerative process is smaller than that of the heat exchange Q_{da} transferred from the regenerator in the other regenerative process. The inadequate heat in the regenerator per cycle can only be compensated from the hot reservoirs in a timely manner, so that the state of the working fluid returns to the original state after each cycle. If not, the temperature of the regenerator would be changed such that the regenerator would not be operated normally. It is thus obvious that a Fermi–Ericsson refrigeration cycle does not possess, in principle, the condition of prefect regeneration and the amount of refrigeration Q_2 per cycle is unvarying because the regenerative losses is compensated from the heat sink at temperature T_{H}.

In order to discuss further the performance of a Fermi–Ericsson refrigeration cycle, we must calculate the times spent on various processes of the cycle. Using eqs (1), (2), (14) and (15), one can obtain the times of the high- and low-temperature isothermal processes as

$$t_1 = \frac{NkT_1[(5/2)F_{a-b} - \ln(z_a/z_b)]}{K_1(T_1 - T_{\mathrm{H}})} \tag{21}$$

and

$$t_2 = \frac{NkT_2[(5/2)F_{d-c} - \ln(z_d/z_c)]}{K_2(T_{\mathrm{L}} - T_2)}, \tag{22}$$

where $F_{a-b} = F(z_a) - F(z_b)$ and $F_{d-c} = F(z_d) - F(z_c)$. In general, the larger the temperature difference of the working fluid in the two isothermal processes, the larger is the amount of regeneration and the longer is the time of the regeneration processes. When the time of regenerative processes is assumed to be directly proportional to the amount of regeneration [26], the time of the two regenerative processes can be given by

$$t_3 + t_4 = \gamma(Q_{bc} + Q_{da}) = \gamma \frac{5}{2} Nk(T_1 F_{a+b} - T_2 F_{c+d}), \tag{23}$$

where γ is a proportional constant independent of temperature, $F_{a+b} = F(z_a) + F(z_b)$ and $F_{c+d} = F(z_c) + F(z_d)$. From eqs (21)–(23), one can calculate the cycle time as

$$\tau = Nk\left\{\frac{T_1[(5/2)F_{a-b} - \ln(z_a/z_b)]}{K_1(T_1 - T_{\mathrm{H}})} + \frac{T_2[(5/2)F_{d-c} - \ln(z_d/z_c)]}{K_2(T_{\mathrm{L}} - T_2)}\right. \\ \left. + \frac{5}{2}\gamma(T_1 F_{a+b} - T_2 F_{c+d})\right\}. \tag{24}$$

4. Optimization on performance parameters

The coefficient of performance, cooling rate and power input are three of the important parameters often considered in the optimal design and theoretical analysis of refrigerators. Using eqs. (15), (18) and (24), we can find that the expressions for the coefficient of performance ε, cooling rate R and power input P are, respectively, given by

$$\varepsilon = \frac{T_2[\ln(z_d/z_c) - (5/2)F_{d-c}]}{T_1 \ln(z_a/z_b) - T_2 \ln(z_d/z_c)}, \tag{25}$$

$$R = \frac{T_2[(5/2)F_{d-c} - \ln(z_d/z_c)]}{\frac{T_1[(5/2)F_{a-b} - \ln(z_a/z_b)]}{K_1(T_1 - T_{\mathrm{H}})} + \frac{T_2[(5/2)F_{d-c} - \ln(z_d/z_c)]}{K_2(T_{\mathrm{L}} - T_2)} + \gamma\frac{5}{2}(T_1 F_{a+b} - T_2 F_{c+d})} \tag{26}$$

and

$$P = \frac{T_1 \ln(z_b/z_a) + T_2 \ln(z_d/z_c)}{\frac{T_1[(5/2)F_{a-b} - \ln(z_a/z_b)]}{K_1(T_1 - T_{\mathrm{H}})} + \frac{T_2[(5/2)F_{d-c} - \ln(z_d/z_c)]}{K_2(T_{\mathrm{L}} - T_2)} + \gamma\frac{5}{2}(T_1 F_{a+b} - T_2 F_{c+d})}. \tag{27}$$

With the help of the above equations, one can optimize these important performance parameters of an irreversible Fermi–Ericsson refrigeration cycle. Using eq. (26), one can plot a three-dimensional diagram (T_1, T_2, R^*) for T_{H}, T_{L}, P_{H}, P_{L}, $K_1 = K_2 = K$, and C, as shown in figure 3, where $C = \gamma K T_{\mathrm{L}}$, $R^* = R/(KT_{\mathrm{L}})$ is the dimensional cooling rate, and the parameters $K = K_1 = K_2$, $P_{\mathrm{L}} = 5 \times 10^5$ Pa, $P_{\mathrm{H}} = 5 \times 10^6$ Pa, $T_{\mathrm{L}} = 8$ K, $T_{\mathrm{H}} = 15$ K, and $C = 0.04$ are chosen. It can be seen from figure 3 that the cooling rate R first increases and then decreases as T_1 or T_2 increases. It shows clearly that there are optimal values of T_1 and T_2 at which the cooling rate R attains its maximum value for a given set of operating parameters. According to eq. (26) and the extreme condition $(\partial R/\partial T_2)_{T_1} = 0$, we can obtain the following equation:

$$\left[\frac{D_1 T_1}{K_1(T_1 - T_{\mathrm{H}})} + \frac{D_2 T_2}{K_2(T_{\mathrm{L}} - T_2)} + \frac{5}{2}\gamma(T_1 F_{a+b} - T_2 F_{c+d})\right] D_3 \\ - D_2 T_2 \left[\frac{D_2 T_{\mathrm{L}} + D_4(T_{\mathrm{L}} - T_2)}{K_2(T_{\mathrm{L}} - T_2)^2} + D_5\right] = 0, \tag{28}$$

where $D_1 = \frac{5}{2}F_{a-b} - \ln(z_a/z_b)$, $D_2 = \frac{5}{2}F_{d-c} - \ln(z_d/z_c)$, $D_3 = \frac{25}{4}F_{d-c} - \frac{9}{4}G_{d-c} - \ln(z_d/z_c)$, $D_4 = \frac{15}{4}F_{d-c} - \frac{9}{4}G_{d-c}$, $D_5 = \frac{1}{2}(3 - 5\gamma)F_{c+d} - \frac{3}{2}G_{c+d}$, $G_{d-c} = \frac{f_{3/2}(z_d)}{f_{1/2}(z_d)} - \frac{f_{3/2}(z_c)}{f_{1/2}(z_c)}$ and $G_{d+c} = \frac{f_{3/2}(z_d)}{f_{1/2}(z_d)} + \frac{f_{3/2}(z_c)}{f_{1/2}(z_c)}$. Equation (28) gives an optimal relation

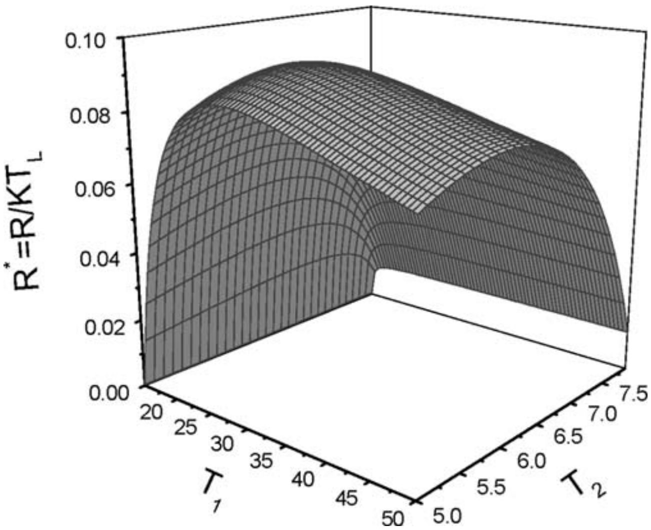

Figure 3. The dimensional cooling rate of the cycle as a function of temperatures (T_1, T_2). The graph is presented for the parameters $T_{\rm H} = 15$ K, $T_{\rm L} = 8$ K, $P_{\rm H} = 5 \times 10^6$ Pa, $P_{\rm L} = 5 \times 10^5$ Pa and $C = 0.04$.

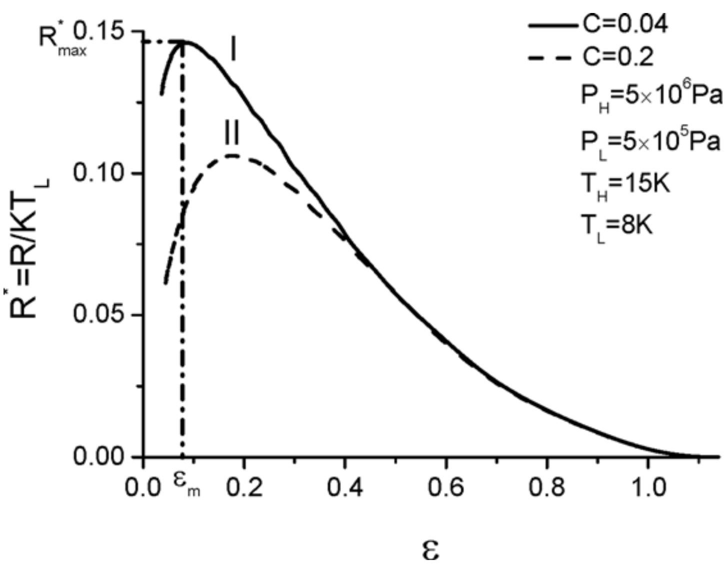

Figure 4. The dimensionless cooling rate $R^* = R/(KT_{\rm L})$ vs. the coefficient of performance ε. Curves I and II are presented for $C = 0.04$ and $C = 0.2$, respectively. The values of the parameters $T_{\rm H}$, $T_{\rm L}$, $P_{\rm H}$ and $P_{\rm L}$ are the same as those used in figure 3.

between T_1 and T_2 for $T_{\rm L}$, $T_{\rm H}$, $P_{\rm L}$, $P_{\rm H}$, K_1, K_2 and γ, but it is too complicated to yield a simple analytical solution. However, for given $T_{\rm L}$, $T_{\rm H}$, $P_{\rm L}$, $P_{\rm H}$, K_1, K_2 and γ, the optimal curves between the cooling rate and coefficient of performance $R^* - \varepsilon$

and the other optimum characteristic curves $P^* - \varepsilon$ and $R^* - P^*$ can be plotted by using eqs (25)–(28) and choosing ^{3}He gas as the ideal Fermi gas, as shown in figures 4–6, where $R^* = R/(KT_{\mathrm{L}})$ and $P^* = P/(KT_{\mathrm{L}})$ are, respectively, the dimensionless cooling rate and power input. Figure 4 shows clearly that the fundamental optimum relation between the cooling rate and the coefficient of performance is not monotonic and there exists a maximum cooling rate $R_{\max}$ and a corresponding coefficient of performance ε_{m} for a set of given parameters K_1, K_2, P_{L}, P_{H}, T_{L}, T_{H} and C. Obviously, for the different given parameters, the maximum cooling rate $R_{\max}$ and the corresponding coefficient of performance ε_{m} will be different. It is also seen from figure 4 that when $R < R_{\max}$, there are two different coefficients of performance for a given cooling rate R, where one is smaller than ε_{m} and the other is larger than ε_{m}. When $\varepsilon < \varepsilon_{\mathrm{m}}$, the cooling rate will decrease as the coefficient of performance is decreased. It is thus clear that the region of $\varepsilon < \varepsilon_{\mathrm{m}}$ is not optimal for a Fermi–Ericsson cryogenic refrigeration cycle. Consequently, the optimal region of the coefficient of performance should be

$$\varepsilon_{\mathrm{m}} \leq \varepsilon < \varepsilon_{\max}, \tag{29}$$

where $\varepsilon_{\max} = T_{\mathrm{L}}\{\ln(z_{d\mathrm{L}}/z_{c\mathrm{L}}) - [(5/2)[F(z_{d\mathrm{L}}) - F(z_{c\mathrm{L}})]\}/[T_{\mathrm{H}}\ln(z_{a\mathrm{H}}/z_{b\mathrm{H}}) - T_{\mathrm{L}}\ln(z_{d\mathrm{L}}/z_{c\mathrm{L}})]$ is the maximum coefficient of performance of an Ericsson cryogenic refrigeration cycle, $z_{a\mathrm{H}} = z(T_{\mathrm{H}}, P_{\mathrm{L}})$, $z_{b\mathrm{H}} = z(T_{\mathrm{H}}, P_{\mathrm{H}})$, $z_{c\mathrm{L}} = z(T_{\mathrm{L}}, P_{\mathrm{H}})$ and $z_{d\mathrm{L}} = z(T_{\mathrm{L}}, P_{\mathrm{L}})$. When a Fermi–Ericsson cryogenic refrigeration cycle is operated in this region, the cooling rate will increase as the coefficient of performance is decreased, and vice versa. It is of significance to note the fact that when $\varepsilon = \varepsilon_{\max}$, the cooling rate of a Fermi–Ericsson cryogenic refrigeration cycle is equal to zero. It shows clearly that the coefficient of performance of a real Fermi–Ericsson cryogenic refrigeration cycle is always smaller than $\varepsilon_{\max}$.

Using the above results and figures 5 and 6, we can further find that the optimal values of the power input should be

$$P \leq P_{\mathrm{m}}, \tag{30}$$

where P_{m} is the power input at the maximum cooling rate. The above results clearly show that the maximum cooling rate $R_{\max}$, coefficient of performance ε_{m} at the maximum cooling rate, power input P_{m} at the maximum cooling rate and maximum coefficient of performance $\varepsilon_{\max}$ are four important performance parameters of a Fermi–Ericsson refrigeration cycle, where $R_{\max}$ and $\varepsilon_{\max}$ determine the upper bounds of the cooling rate and coefficient of performance and ε_{m} and P_{m} determine the allowable values of the lower and upper bounds of the optimal coefficient of performance and power input, respectively.

5. Several interesting gases

It is significant to note that for some special cases, the results obtained above may be simplified.

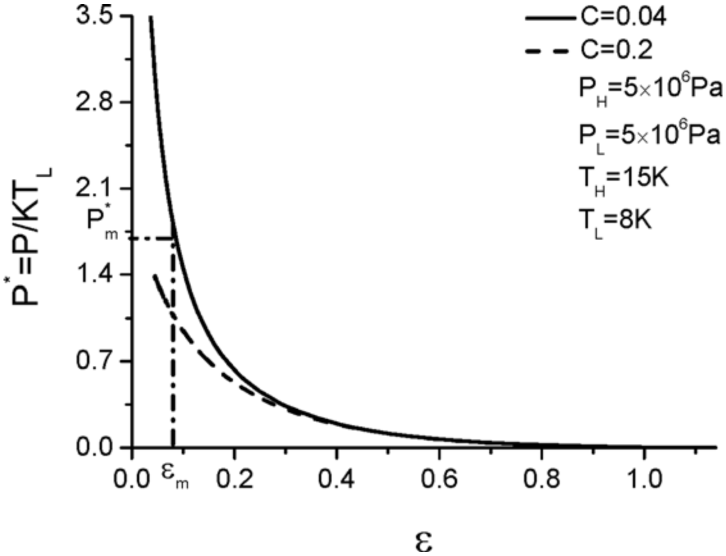

Figure 5. The dimensionless power input $P^* = P/(KT_\mathrm{L})$ vs. the coefficient of performance ε. The values of the parameters C, T_H, T_L, P_H and P_L are the same as those used in figure 3.

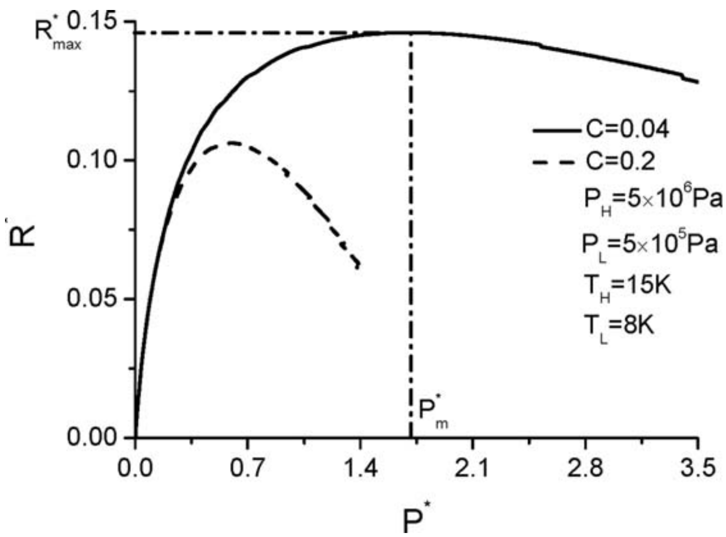

Figure 6. The dimensionless cooling rate $R^* = R/(KT_\mathrm{L})$ vs. the dimensionless power input $P^* = P/(KT_\mathrm{L})$. The values of the parameters C, T_H, T_L, P_H and P_L are the same as those used in figure 3.

5.1 *Strong gas degeneracy*

Under the very low-temperature and high-density condition, i.e., the condition of strong gas degeneracy, according to Sommerfeld's lemma, the function $f_n(z)$ can be expressed as asymptotic expansions in power of $(\ln z)^{-1}$. In the first approximation, the Fermi–Dirac function $F(T, P)$ and natural logarithm of the fugacity $\ln z$ can be, respectively, given by [27]

$$F(T,P) = \frac{2}{5}\frac{T_{\mathrm{F}}(P)}{T} + \frac{\pi^2}{10}\frac{T}{T_{\mathrm{F}}(P)} \tag{31}$$

and

$$\ln z = \frac{T_{\mathrm{F}}(P)}{T} - \frac{\pi^2}{4}\frac{T}{T_{\mathrm{F}}(P)}, \tag{32}$$

where $T_{\mathrm{F}}(P) = AP^{2/5}$ is the Fermi temperature and $A = (15\pi^2\hbar^3)^{2/5}/(2km^{3/5})$. Using eqs (31) and (32), eqs (19), (25)–(27) can be, respectively, simplified as

$$\Delta Q = Nk\frac{\pi^2}{4A}(T_1^2 - T_2^2)(P_{\mathrm{H}}^{-2/5} - P_{\mathrm{L}}^{-2/5}) < 0, \tag{33}$$

$$\varepsilon = \frac{2T_2^2}{T_1^2 - T_2^2} = \frac{T_2}{T_1 - T_2}\frac{2T_2}{T_1 + T_2} < \varepsilon_{\mathrm{EC}}, \tag{34}$$

$$R = \left\{\frac{(T_1/T_2)^2}{K_1(T_1 - T_{\mathrm{H}})} + \frac{1}{K_2(T_{\mathrm{L}} - T_2)} + \frac{\gamma}{2}\frac{r_{\mathrm{P}}^{2/5} + 1}{r_{\mathrm{P}}^{2/5} - 1}\left[(T_1/T_2)^2 - 1\right]\right\}^{-1}, \tag{35}$$

and

$$P = \left[\frac{2T_1^2}{K_1(T_1 - T_{\mathrm{H}})(T_1^2 - T_2^2)} + \frac{2T_2^2}{K_2(T_{\mathrm{L}} - T_2)(T_1^2 - T_2^2)} + \gamma\frac{r_{\mathrm{P}}^{2/5} + 1}{r_{\mathrm{P}}^{2/5} - 1}\right]^{-1}, \tag{36}$$

where $\varepsilon_{\mathrm{EC}} = T_2/(T_1 - T_2)$ is the coefficient of performance of an endoreversible Carnot refrigeration cycle and $r_{\mathrm{P}} = P_{\mathrm{H}}/P_{\mathrm{L}}$ is the pressure ratio. It can be seen from eq. (34) that in the case of strong gas degeneracy, the coefficient of performance of a Fermi–Ericsson refrigeration cycle is only a function of temperature and independent of other parameters. By using eqs (34) and (35), the cooling rate can be further expressed as

$$R = \left[\frac{2\varepsilon^{-1} + 1}{K_1(\sqrt{2\varepsilon^{-1} + 1}T_2 - T_{\mathrm{H}})} + \frac{1}{K_2(T_{\mathrm{L}} - T_2)} + \gamma\frac{1}{\varepsilon}\frac{r_{\mathrm{P}}^{2/5} + 1}{r_{\mathrm{P}}^{2/5} - 1}\right]^{-1}. \tag{37}$$

Using eq. (37) and the external condition $(\partial R/\partial T_2)_\varepsilon = 0$, we can find that the fundamental optimal relations between some important parameters and the coefficient of performance are, respectively, given by

$$R = \left\{ \frac{[\sqrt{K_1}(2\varepsilon^{-1}+1)^{1/4} + \sqrt{K_2}(2\varepsilon^{-1}+1)^{1/2}]^2}{K_1 K_2 T_\mathrm{L}[(2\varepsilon^{-1}+1)^{1/2} - T_\mathrm{H}/T_\mathrm{L}]} + \gamma \frac{1}{\varepsilon} \frac{r_\mathrm{P}^{2/5}+1}{r_\mathrm{P}^{2/5}-1} \right\}^{-1}, \tag{38}$$

$$P = \frac{1}{\varepsilon} \left\{ \frac{[\sqrt{K_1}(2\varepsilon^{-1}+1)^{1/4} + \sqrt{K_2}(2\varepsilon^{-1}+1)^{1/2}]^2}{K_1 K_2 T_\mathrm{L}[(2\varepsilon^{-1}+1)^{1/2} - T_\mathrm{H}/T_\mathrm{L}]} + \gamma \frac{1}{\varepsilon} \frac{r_\mathrm{P}^{2/5}+1}{r_\mathrm{P}^{2/5}-1} \right\}^{-1}, \tag{39}$$

$$T_1 = \frac{\sqrt{K_1/K_2}T_\mathrm{H} + (2\varepsilon^{-1}+1)^{3/4}T_\mathrm{L}}{\sqrt{K_1/K_2} + (2\varepsilon^{-1}+1)^{1/4}} \tag{40}$$

and

$$T_2 = \frac{\sqrt{K_1/K_2}(2\varepsilon^{-1}+1)^{-1/2}T_\mathrm{H} + (2\varepsilon^{-1}+1)^{1/4}T_\mathrm{L}}{\sqrt{K_1/K_2} + (2\varepsilon^{-1}+1)^{1/4}}. \tag{41}$$

It is clearly seen from eq. (38) that the cooling rate R is zero when $\varepsilon = 0$ or $\varepsilon = 2T_\mathrm{L}^2/(T_\mathrm{H}^2 - T_\mathrm{L}^2) = \varepsilon_\mathrm{max}$. This implies the fact that when the coefficient of performance is equal to some value, there is a maximum for the cooling rate, as shown in figures 7–9. It can be seen from figures 7–9 that for different given parameters, the maximum cooling rate R_max and the corresponding coefficient of performance ε_m are different. For example, for the given parameters r_P and τ, the larger the parameter C, the smaller is the maximum cooling rate R_max while the corresponding coefficient of performance ε_m will be larger; for the given parameters C and τ, the larger the pressure ratio r_P, the larger is the maximum cooling rate R_max while the corresponding coefficient of performance ε_m will be smaller; for given parameters C and r_P, the smaller the temperature τ, the larger is the maximum cooling rate R_max and the corresponding coefficient of performance ε_m.

Comparing figure 7 with figure 4, one can find that the stronger the quantum degeneracy of the working fluid, the smaller is the maximum cooling rate R_max and the coefficient of performance ε_max while the coefficient of performance ε_m at the maximum cooling rate will be larger. Obviously, it is distinct that the influence of quantum degeneracy of the working fluid on the performance and optimal performance of a low-temperature Fermi–Ericsson refrigeration cycle. This shows that it is necessary to consider the effect of quantum degeneracy on the performance of a thermodynamic cycle at low temperatures.

5.2 *Weak gas degeneracy*

Under the higher temperature or lower-density condition, i.e., the condition of weak gas degeneracy, $0 < z < 1$ and the Fermi–Dirac function $f_n(z)$ may be expanded in power of z, i.e.

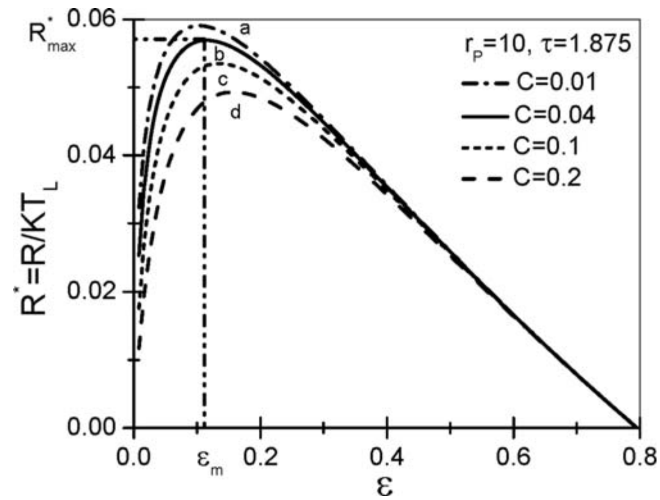

Figure 7. The dimensionless cooling rate $R^* = R/(KT_L)$ vs. the coefficient of performance ε curves for given parameters $r_P = 10$ and $\tau = 1.875$. Curves a, b, c and d correspond to the cases $C = 0.01, 0.04, 0.1$ and 0.2, respectively.

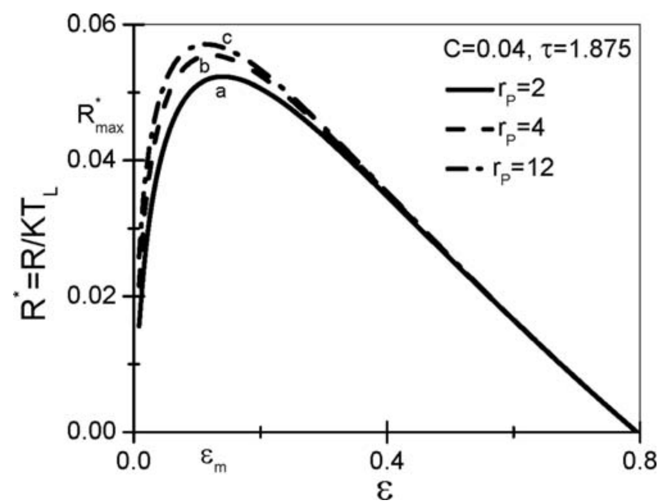

Figure 8. The dimensionless cooling rate $R^* = R/(KT_L)$ vs. the coefficient of performance ε curves for given parameters $C = 0.04$ and $\tau = 1.875$. Curves a, b and c correspond to the cases $r_P = 2, 4$ and 12, respectively.

$$f_n(z) = \sum_{l=1}^{\infty} (-1)^{l-1} \frac{z^l}{l^n}. \tag{42}$$

The function $F(T, P)$ and fugacity z may be expanded to the first approximation as [27]

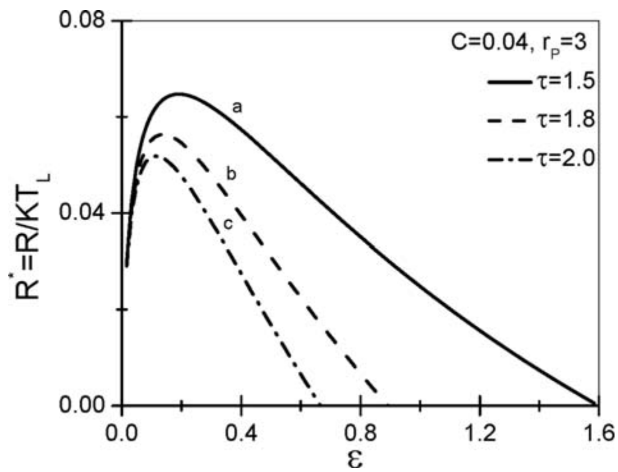

Figure 9. The dimensionless cooling rate $R^* = R/(KT_\mathrm{L})$ vs. the coefficient of performance ε curves for given parameters $C = 0.04$ and $r_\mathrm{P} = 10$. Curves a, b and c correspond to the cases $r_\mathrm{P} = 2, 4$ and 12, respectively.

$$F(T,P) = 1 + \frac{BP}{T^{5/2}} \tag{43}$$

and

$$z = \frac{4\sqrt{2}BP}{T^{5/2}}(1 + BP/T^{5/2}), \tag{44}$$

where $B = h^3/[4\sqrt{2}gk^{5/2}(2\pi m)^{3/2}]$. By using eqs (43) and (44), eqs (19) and (25)–(28) can be, respectively, simplified as

$$\Delta Q = \tfrac{5}{2}NkB(P_\mathrm{H} - P_\mathrm{L})(T_1^{-3/2} - T_2^{-3/2}) < 0, \tag{45}$$

$$\varepsilon = \frac{T_2 - \frac{3}{2}BT_2^{-3/2}\Delta P/\ln r_\mathrm{P}}{T_1 - T_2 + B\Delta P(T_1^{-3/2} - T_2^{-3/2})/\ln r_\mathrm{P}}, \tag{46}$$

$$R = \frac{T_2 \ln r_\mathrm{P} - \frac{3}{2}BT_2^{-3/2}\Delta P}{\left\{\frac{T_1 \ln r_\mathrm{P} - \frac{3}{2}BT_1^{-3/2}\Delta P}{K_1(T_1 - T_\mathrm{H})} + \frac{T_2 \ln r_\mathrm{P} - \frac{3}{2}BT_2^{-3/2}\Delta P}{K_2(T_\mathrm{L} - T_2)} + \frac{5}{2}\gamma[2(T_1 - T_2) + B\Sigma P(T_1^{-3/2} - T_2^{-3/2})]\right\}}, \tag{47}$$

$$P = \frac{(T_1 - T_2)\ln r_\mathrm{P} + B\Delta P(T_1^{-3/2} - T_2^{-3/2})}{\left\{\frac{T_1 \ln r_\mathrm{P} - \frac{3}{2}BT_1^{-3/2}\Delta P}{K_1(T_1 - T_\mathrm{H})} + \frac{T_2 \ln r_\mathrm{P} - \frac{3}{2}BT_2^{-3/2}\Delta P}{K_2(T_\mathrm{L} - T_2)} + \frac{5}{2}\gamma[2(T_1 - T_2) + B\Sigma P(T_1^{-3/2} - T_2^{-3/2})]\right\}}, \tag{48}$$

and

$$\left(\ln r_{\mathrm{P}} + \frac{9}{4} B\Delta P T_2^{-5/2}\right) \left\{ \frac{T_1 \ln r_{\mathrm{P}} - \frac{3}{2} B\Delta P T_1^{-3/2}}{K_1(T_1 - T_{\mathrm{H}})} \right.$$
$$\left. + \frac{5}{2}\gamma \left[2(T_1 - T_2) + B\Sigma P (T_1^{-3/2} - T_2^{-3/2}) \right] \right\}$$
$$- \left(T_2 \ln r_{\mathrm{P}} - \frac{3}{2} B\Delta P T_2^{-3/2}\right) \left[\frac{T_2 \ln r_{\mathrm{P}} - \frac{3}{2} B\Delta P T_2^{-3/2}}{K_2(T_{\mathrm{L}} - T_2)^2} \right.$$
$$\left. + \frac{5}{2}\gamma \left(\frac{3}{2} B\Sigma P T_2^{-5/2} - 2 \right) \right] = 0, \tag{49}$$

where $\Delta P = P_{\mathrm{H}} - P_{\mathrm{L}}$ and $\Sigma P = P_{\mathrm{H}} + P_{\mathrm{L}}$. Using eqs (46)–(49), one can discuss in detail the optimal performance of an irreversible Fermi–Ericsson refrigeration cycle in the weak gas degeneracy case.

5.3 *At high temperatures*

When the temperature of the working fluid is high enough and its density is low enough, the fugacity of the Fermi gas z is much smaller than unity. In such a case, $F(z) = 1$ and $f_n(z) = z$. Equations (19) and (25)–(27) can be, respectively, simplified as

$$\Delta Q = 0, \tag{50}$$

$$\varepsilon = \frac{T_2}{T_1 - T_2}, \tag{51}$$

$$R = \frac{T_2}{T_1/[K_1(T_1 - T_{\mathrm{H}})] + T_2/[K_2(T_{\mathrm{L}} - T_2)] + 5\gamma(T_1 - T_2)/\ln r_{\mathrm{P}}} \tag{52}$$

and

$$P = \frac{T_1 - T_2}{T_1/[K_1(T_1 - T_{\mathrm{H}})] + T_2/[K_2(T_{\mathrm{L}} - T_2)] + 5\gamma(T_1 - T_2)\ln r_{\mathrm{P}}}. \tag{53}$$

Substituting eq. (51) into eq. (52) gives

$$R = \left\{ \frac{1 + \varepsilon^{-1}}{K_1[(1 + \varepsilon^{-1})T_2 - T_{\mathrm{H}}]} + \frac{1}{K_2(T_{\mathrm{L}} - T_2)} + 5\gamma/(\varepsilon \ln r_{\mathrm{P}}) \right\}^{-1}. \tag{54}$$

Using eq. (54) and the external condition $(\partial R/\partial T_2)_\varepsilon = 0$, one can prove that the fundamental optimum relation between the cooling rate and the coefficient of performance of an irreversible Fermi–Ericsson refrigeration cycle at high temperatures is given by

$$R = \frac{K_{12}T_\mathrm{L}\varepsilon(\varepsilon_\mathrm{c} - \varepsilon)}{\varepsilon_\mathrm{c}\varepsilon(1+\varepsilon) + a_1 K_{12}T_\mathrm{L}(\varepsilon_\mathrm{c} - \varepsilon)}, \tag{55}$$

where $K_{12} = \frac{K_1 K_2}{(\sqrt{K_1}+\sqrt{K_2})^2}$, $a_1 = 5\gamma/\ln r_\mathrm{P}$ and $\varepsilon_\mathrm{c} = T_\mathrm{L}/(T_\mathrm{H} - T_\mathrm{L})$ is the coefficient of performance of a reversible Carnot refrigerator. It is of interest to compare the results obtained here with those derived from a classical Ericsson or Stirling refrigeration cycle working with an ideal gas. It can be found that when the influence of finite-rate heat transfer between the working fluid and the heat reservoirs on the optimal performance of the classical Ericsson or Stirling refrigeration cycle is considered and the heat transfer is assumed to obey a linear law, the fundamental optimum relation between the cooling rate and the coefficient of performance of the Fermi–Ericsson refrigeration cycle in the high temperature limit is the same as that of the classical Ericsson refrigeration cycle. When C_P is replaced by the heat capacity at constant volume C_V, one can directly obtain the fundamental optimum relation of the classical Stirling refrigeration cycle. This is just an expected result because the quantum behavior of gas particles in this case is negligible and the refrigeration cycle may posses the condition of perfect regeneration.

6. Conclusions

An important cycle model of the irreversible quantum Ericsson refrigeration cycle using an ideal Fermi gas as the working fluid has been established. On the basis of the theory of the statistical mechanics and thermodynamics, we have analyzed the influence of both the quantum degeneracy of the working fluid and the irreversibility of the finite-rate heat transfer between the working fluid and the heat reservoirs on the performance characteristics of the quantum Ericsson refrigeration cycle. The general expressions for several important performance parameters, such as the coefficient of performance, cooling rate, power input, cyclic period and inherent regenerative losses, are derived. By using the expressions, the effect of non-perfect regeneration is analyzed and the optimal relation between the cooling rate and coefficient of performance is obtained. Several optimal performance characteristic curves are generated. The optimally operating regions of the cryogenic refrigeration cycle are determined and the optimum criteria of some important performance parameters are given. The optimal performance characteristics of the cycle in strong and weak gas degeneracy cases are discussed in detail. It can be clearly seen from these optimal characteristics that the optimal cooling rate and power input, in general, is dependent not only on temperatures of the heat reservoirs and the thermal conductances between the working fluid and the heat reservoirs but also on the pressures of two isobaric processes and other parameters. Because of the influence of the quantum degeneracy, the cooling rate of an irreversible Fermi–Ericsson refrigeration cycle is always less than that of an irreversible classical Ericsson refrigeration cycle working with an ideal gas for the same given coefficient of performance. The maximum cooling rate decreases as the quantum effects of the gas increases, while the coefficient of performance at the maximum cooling rate is increasable. Moreover, it is pointed out that in the high-temperature limit, the Fermi–Ericsson refrigeration cycle becomes the classical Ericsson refrigeration cycle, so that the optimal performance of the classical Ericsson or Stirling refrigeration cycles investigated widely in literature may be directly derived from the results in the present article. The results obtained here will be helpful to further understand the general performance characteristics of the Fermi–Ericsson refrigeration cycle.

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